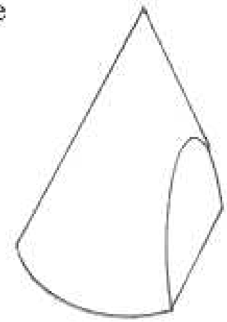
Volume Removed From Truncated Cone



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Problem: Given a right circular cone, let a plane parallel to its axis slice off a portion of the cone (less than half the cone). Determine the volume of the removed portion.

Pedagogical Value: This exercise has the intrinsic interest of a serious, real-world problem (see below). It combines a variety of tasks, some or all of which may be assigned:

(1) Visualization of a solid.

(2) Locating the solid in a coordinate system and labeling relevant distances.

(3) Setting up the integral for the volume and realizing that cylindrical coordinates are appropriate.

(4) Evaluating the integral.

(5) Carrying out a reality check by seeing whether the value obtained tends to the expected values of zero or of half a cone's volume as the slice moves toward the edge of the cone or toward its axis.

Background to the problem. In 1991 a former student, employed by a prominent engineering firm, sent me this problem. It arose because a set of steel girders shorter than specified had been installed at a major project*. In order to determine whether, with the shorter girders, the structure would still be sound, the engineers needed to know the volume of the missing part of a cone. (This cone modeled the strength of a bolt tying together three such girders.) His engineering colleagues could not figure out the volume, so he sent it to me.

* "... Let me give you some background on the problem. One of our current projects at ... involves the reconstruction of the ... Elevated Line in ... This is one branch of the ... subway which operates underground in the center city and is carried on elevated viaducts further out. The viaducts are quite old and in disrepair. We have designed replacement structures and are overseeing the construction of this project.

Well, it seems that one of our contractors . . . used some steel girders that were shorter than the contract specified. Picture, if you will, three metal girders joined by a bolt (the girders for the sides of the superstructure on which the rails rest). The bolt is held in place by a large nut and washer. My structural engineer friends tell me that the force placed by the nuts and bolt on the girders (the force that holds them together) is distributed in a conical shape, with its tip at the center of the top of the bolt, and its base between the girders. The problem was that the top girder was not long enough, thus truncating the critical cone area distributing the force. So my structural friends needed to determine the volume of the missing area in order to develop an adjustment factor and calculate the force that would be holding the girders together. Based on the results of that calculation, the structure will either be determined to be sound with the shorter girders, or the contractor will be required to replace the offending girders with others meeting the contract specifications. How's that for a real life application - they're trying to figure out whether it will fall down or not!"

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                         _______
                        | | | |  <--Top of bolt
                        |_|_|_|
                     ___=======  <--Washer________________
                    |   /|     |\           Upper girder
 missing            |  / |     | \
 part of            | /  |     |  \<-- truncated cone
 cone------>        |    |     |   \
 ___________________|____|_____|____\_____________________
|                        |     |            Lower girder
|                        |     |
|                        |     |<-- the bolt
|                        |     |
|________________________|_____|__________________________
|                        |     |            Lowest girder
|                        |     |
          ...              ...          ...
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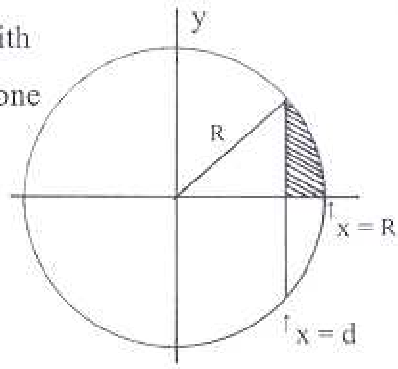

Solution: Consider a right circular cone of height h, base radius R, with the base a circular disk in the x-y plane, centered at the origin. Slice the cone perpendicular to the x-y plane, along the line $x = d$, with $0 < d < R$. The height z on the cone's surface is given by $z = h - \frac{h}{R}\sqrt{x^2 + y^2}$. The truncated portion of the cone, whose volume we seek, is that portion above the segment S of the base, consisting of those points which lie to the right of $x = d$. We compute this volume by doubling the volume above the upper half-plane portion of S, i.e. above the shaded points in the base, for which both $x \ge d$ and $y \ge 0$. We use polar coordinates, so the line $x = d$ becomes $r = \frac{d}{\cos\theta}$. For each fixed θ, the relevant points in S have radial values r satisfying $\frac{d}{\cos\theta} \le r \le R$. The angular values θ of points in S satisfy $-\cos^{-1}\frac{d}{R} \le \theta \le \cos^{-1}\frac{d}{R}$. Therefore the volume in question can be expressed as

$$2\int_0^{\cos^{-1}\frac{d}{R}} \int_{\frac{d}{\cos\theta}}^{R} (h - \frac{h}{R}r)r\,dr\,d\theta \;=\; 2h\int_0^{\cos^{-1}\frac{d}{R}} \int_{\frac{d}{\cos\theta}}^{R} (r - \frac{1}{R}r^2)\,dr\,d\theta \;=\; 2h\int_0^{\cos^{-1}\frac{d}{R}} (\frac{r^2}{2} - \frac{r^3}{3R})\Big|_{d\sec\theta}^{R}\,d\theta \;=$$

$$2h(\frac{R^2}{2} - \frac{R^3}{3R})\cos^{-1}\tfrac{d}{R} \;-\; 2h\int_0^{\cos^{-1}\frac{d}{R}} (\tfrac{1}{2}d^2\sec^2\theta - \tfrac{1}{3R}d^3\sec^3\theta)\,d\theta \;=$$

$$\frac{hR^2}{3}\cos^{-1}\tfrac{d}{R} - hd^2\tan(\cos^{-1}\tfrac{d}{R}) + \frac{2hd^3}{3R}\int_0^{\cos^{-1}\frac{d}{R}} \sec^3\theta\,d\theta \;=\; \text{(by parts or via tables)}$$

$$\frac{hR^2}{3}\cos^{-1}\tfrac{d}{R} - hd^2\frac{\sqrt{R^2 - d^2}}{d} + \frac{2hd^3}{3R}\left(\frac{1}{2}\sec(\theta)\tan(\theta) + \frac{1}{2}ln\left|\sec(\theta) + \tan(\theta)\right|\right)\Bigg|_0^{\cos^{-1}\frac{d}{R}} =$$

$$\frac{hR^2}{3}\cos^{-1}\tfrac{d}{R} - hd\sqrt{R^2 - d^2} + \frac{hd^3}{3R}\left(\sec(\cos^{-1}\tfrac{d}{R})\tan(\cos^{-1}\tfrac{d}{R}) + ln\left|\sec(\cos^{-1}\tfrac{d}{R}) + \tan(\cos^{-1}\tfrac{d}{R})\right|\right) =$$

$$\frac{hR^2}{3}\cos^{-1}\tfrac{d}{R} - hd\sqrt{R^2 - d^2} + \frac{hd^3}{3R}\left(\frac{R}{d}\frac{\sqrt{R^2 - d^2}}{d} + ln\left|\frac{R}{d} + \frac{\sqrt{R^2 - d^2}}{d}\right|\right).$$

Thus the volume is: $\boxed{\frac{hR^2}{3}\cos^{-1}\tfrac{d}{R} - \frac{2}{3}hd\sqrt{R^2 - d^2} + \frac{hd^3}{3R}ln\left(R + \sqrt{R^2 - d^2}\right) - \frac{hd^3}{3R}ln\,d}$.

For a reality check on the above closed form solution, we first note that as $d \to R$, the volume tends to $\frac{hR^2}{3}\cos^{-1}1 - 0 + \frac{hR^3}{3R}ln\,R - \frac{hR^3}{3R}ln\,R = 0$. Next, let $d \to 0+$. L'Hôpital's Rule shows that the final term, $\frac{hR^3}{3R}ln\,R$, goes to zero. The middle two terms also go to zero, and the first term tends, appropriately, to $\frac{\pi hR^2}{6}$, i.e. half the volume of the cone.