Finding a Remote Root to a Transcendental Equation

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Problem: To find the (approximate) solutions to $\frac{3620}{x} + \ln x = 16.82$.

Pedagogical Value: This exercise has the intrinsic interest of being a real question encountered in professional work (see below). It has several facets, any of which can be a basis for an assignment:

(1) Checking the claimed solution. The utility of a basic reality check is worth instilling.

(2) Figuring out what to do when the claimed solution is slightly wrong. This gets into matters of numerical analysis and approximation, an area of which students should be made aware, even though calculus is not the course where details of such work are appropriate.

(3) Recognizing that there is a second solution is a useful exercise in the methods of first semester calculus.

(4) Realizing that a graphing calculator or program is unlikely to suggest the presence of a second solution. This is a useful lesson in the value of an analytic approach.

(5) Both the question of whether there is a second solution and obtaining a reasonable approximation to it are good lessons in the slow growth of the logarithm and the rapid growth of exponentials.

Background to the problem. In 1995 a former student, then in a graduate program in Materials Science, wrote, "How does one solve: $-\frac{3620}{T} - \ln T = -16.82$?

We know the answer is 328, but we're not sure what the way to solve this is. Thanks."

Solution: For simplicity we consider the equivalent $\frac{3620}{T} + \ln T = 16.82$. A closed form solution is out of the question, but 328 is a crude estimate of one root – it's the nearest integer to the smaller of two solutions. As such, it just misses yielding 16.82:

With $f(T) = \frac{3620}{T} + \ln T, \quad f(328) = 16.83.$

Because of the slow growth of the logarithm, graphing calculators or programs are unlikely to show that $f(T)$ goes to ∞ as T grows, unless the user recognizes that this must be the case and adjusts the instruments used. On the other hand, computer algebra systems or calculators with equation solvers should readily find the two solutions. (To six digits, these are 328.314 and 2.01723×10^7. Such precision is probably unwarranted, because the 16.82 value is itself likely to be an approximation.) In what follows I explain a way of approximating the second solution.

When discussing rates of growth in second semester calculus, I ask students, after noting the first solution, the following:

> "Find a second solution T for which $\frac{3620}{T} + ln(T$ differs from 16.82 by no more than
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> and explain why your result satisfies the specified tolerance. Note that the task can be dramatically simplified if you make effective use of the rate of growth of the logarithm function."

For $f(T) = \frac{3620}{T} + lnT$, note that $f'(T) = -\frac{3620}{T^2} + \frac{1}{T}$, so f decreases on $(0, 3620]$ to its minimum value, which is about 9.2. It increases (to ∞) for $T \geq 3620$. Thus there are exactly two roots of $f(T) = 16.82$. To find the larger root, first take the exponential of each side of $\frac{3620}{T} + lnT = 16.82$, obtaining $\left(e^{3620/T}\right)T = e^{16.82}$.

Next, note that any value of $T > 3620$ for which $\left(e^{3620/T}\right)T = e^{16.82}$ will have to satisfy $e^1T = \left(e^{3620/3620}\right)T > \left(e^{3620/T}\right)T = e^{16.82}$, so T will be very large. For such large T, $\frac{3620}{T}$ will be approximately 0, suggesting that instead of solving $\left(e^{3620/T}\right)T = e^{16.82}$ we consider solving $T = e^{16.82}$ This latter is feasible: $T = 20{,}175{,}912.49$

Finally, observe that $ln\,(20{,}175{,}912.49) = 16.81996$, to five places after the decimal and $\frac{3{,}620}{20{,}175{,}912.49} = .00017942$ to five significant digits. Thus $T = 20{,}175{,}912.49$ does the job. slightly simpler $20{,}175{,}912.5$ also satisfies the conditions.

Remarks Because of the large numbers involved, many graphing devices or programs may fail to detect or even suggest second solution, unless special care is taken when experimenting with them. The slow growth of the logarithm function is what forces the large size of the second solution. This characteristic of logarithmic growth is not always appreciated, as shown by Ed Barbeau's column "Fallacies, Flaws, and Flimflam" in the November, 2005 *College Math. Journal* (Vol. 36, #5, pp. 394-396.)